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η-Weak-Pseudo-Hermiticity Generators and Radially Symmetric Hamiltonians

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Abstract A class *η-weak-pseudo-Hermiticity generators* for spherically symmetric non-Hermitian Hamiltonians are presented. An operators-based procedure is introduced so that the results for the 1D Schrödinger Hamiltonian may very well be reproduced. A generalization beyond the nodeless states is proposed. Our illustrative examples include *η-weak-pseudo-Hermiticity generators* for the non-Hermitian weakly perturbed 1D and radial oscillators, and the non-Hermitian perturbed radial Coulomb.

Keywords Pseudo-Hermiticity · Radial Schrödinger equation · η-weak-pseudo-Hermiticity generators

1 Introduction

The consensus that "*the existence of real spectrum need not necessarily be attributed to the Hermiticity of the Hamiltonian*" has offered a sufficiently strong motivation for the continued interest in the complex, non-Hermitian, Hamiltonians [1–47]. Intensive studies on such Hamiltonians resulted in the proposal of the $\mathcal{PT}$-symmetric quantum mechanics by Bender and Boettcher [7], where the Hamiltonian Hermiticity assumption $H = H^{\dagger}$ is replaced by the mere $\mathcal{PT}$-symmetry, where $\mathcal{P}$ denotes the parity ($\mathcal{P}x\mathcal{P} = -x$) and the anti-linear operator $\mathcal{T}$ mimics the time reflection ($\mathcal{T}i\mathcal{T} = -i$). Among the first $\mathcal{PT}$-symmetric models with physically acceptable impact has been the Buslaev and Grecchi quartic anharmonic oscillator [1–6] described by the radial Schrödinger equation

$$\left\{-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r)\right\}\psi(r) = E\psi(r), \tag{1}$$

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where, in the presence of a less traditional context, a coordinate shift to the complex plane, $r \to x - ic$, is introduced (with real $x \in (-\infty, \infty)$ and a constant $\mathrm{Im}\, r = -c < 0$) and $\psi(r) \in L_2(-\infty, \infty)$ is required. For explicit illustration the reader may refer to, e.g., Znojil and Lévai in [3].

However, subsequent recent studies emphasized that the $\mathcal{PT}$-symmetric Hamiltonians constitute a subclass of a very broader class of the so-called pseudo-Hermiticity of these non-Hermitian Hamiltonians [26–37]. A Hamiltonian H is pseudo-Hermitian if it obeys the similarity transformation:

$$\eta H \eta^{-1} = H^{\dagger}, \tag{2}$$

where η is a Hermitian (and so is ηH) invertible linear operator and (†) denotes the adjoint. In such settings, it is concreted (cf., e.g. [29, 30, 35–37]) that H, with a complete biorthonormal eigenvectors, has a real spectrum and is *η-pseudo-Hermitian* with respect to the nontrivial "metric" operator

$$\eta = O^{\dagger} O, \tag{3}$$

where O is some linear invertible operator $O : \mathcal{H} \to \mathcal{H}$, and $\mathcal{H}$ is the Hilbert space of the quantum system with a Hamiltonian H and infinitely many η satisfying (2) (cf., e.g., [26–28, 37–41]). Moreover, an *η-pseudo-Hermitian* Hamiltonian equivalently satisfies the *intertwining* relation:

$$\eta H = H^{\dagger} \eta. \tag{4}$$

However, one may relax H to be *η-weak-pseudo-Hermitian* by not restricting the intertwining *second-order* differential operator η to be Hermitian (cf., e.g., Bagchi and Quesne [40]), linear and/or invertible (cf. e.g., Solombrino [39]). Yet, without enforcing invertibility as a necessary condition on O and hence on η, Fityo [38] has implicitly used *η-weak-pseudo-Hermiticity* and constructed 1D non-Hermitian *η-weak-pseudo-Hermitian Hamiltonians* via 1D *η-weak-pseudo-Hermiticity generators*. Moreover, very recently we have (Mustafa and Mazharimousavi [41]) considered the nodeless states of some non-Hermitian d-dimensional Hamiltonians with position-dependent mass and their *η-(weak)-pseudo-Hermiticity generators*. We have also explored [38–41] exact solvability as a byproduct of such generators in 1D.

In this work, we follow Fityo's [38] (as well as our work in [38–41]) *η-weak-pseudo-Hermiticity* condition (i.e., not enforcing invertibility condition on O and hence on η) and generalize it to present a class of spherically symmetric non-Hermitian *η-weak-pseudo-Hermitian Hamiltonians* via their *η-weak-pseudo-Hermiticity generators*, for multi-nodal states. An operators-based procedure is introduced (in Sect. 2) so that the results of Fityo's formalism [38–41], for the one-dimensional (1D) Schrödinger Hamiltonian, may very well be reproduced. On this issue, the reader may be reminded that one can safely return back from the redial Schrödinger equation (1) to the one dimensional case by the trivial choice $\ell = -1$ and/or $\ell = 0$ (cf., e.g., the sample in [22–25]). Our illustrative examples, in Sect. 3, include *η-weak-pseudo-Hermiticity generators* for the non-Hermitian weakly perturbed 1D and radial oscillators, and the non-Hermitian perturbed radial Coulomb models. Therein, we present not only nodeless radial wave functions but also some of the multi-nodal ones. We give our concluding remarks in Sect. 4.

2 η-Weak-Pseudo-Hermiticity Generators and Radial Symmetry; A Second-Order Intertwining η

Consider a class of spherically symmetric non-Hermitian Hamiltonians (in $\hbar = 2m = 1$ units) of the form

$$H = -\partial_r^2 + V_{eff}(r); \qquad V_{eff}(r) = \frac{\ell(\ell+1)}{r^2} + V(r) + iW(r), \tag{5}$$

where $V(r)$ and $W(r)$ are real-valued radial functions, and ℓ is the angular momentum quantum number. Then H has a real spectrum if there is a linear operator $O : \mathcal{H} \to \mathcal{H}$ such that H is an η-*weak-pseudo-Hermitian Hamiltonian* satisfying the intertwining relation (4). With the linear operator

$$O = \partial_r + Z(r) \quad \Longrightarrow \quad O^\dagger = -\partial_r + Z^*(r) \tag{6}$$

where

$$Z(r) = F(r) + iG(r), \quad F(r) = -(\ell+1)/r + f(r) \tag{7}$$

and $F(r)$ and $G(r)$ are real-valued radially symmetric functions and $\mathbb{R} \ni r \in (0, \infty)$, equation (3) implies

$$\eta = -\partial_r^2 + M(r)\partial_r + N(r), \tag{8}$$

where $M(r) = Z^*(r) - Z(r)$, $N(r) = Z^*(r)Z(r) - Z'(r)$, and primes denote derivatives with respect to r. Herein, it should be noted that the operators O and $O^\dagger$ are two intertwining operators and the Hermitian operator η only plays the role of a certain auxiliary transformation of the dual Hilbert space and leads to the intertwining relation (4) (cf., e.g., [26–30, 42]). Hence, using relation (4) along with the eigenvalue equation for the Hamiltonian, $H/E_i\rangle = E_i/E_i\rangle$, and its adjoint, $H^\dagger/E_i\rangle = E_i^*/E_i\rangle$, one can show that any two eigenvectors of H satisfy (cf., e.g., Mostafazadeh in [26])

$$\langle E_i/H^\dagger\eta - \eta H/E_j\rangle = 0 \quad \Longrightarrow \quad \left(E_i^* - E_j\right)\langle\langle E_i/E_j\rangle\rangle_\eta = 0, \tag{9}$$

which implies that if $E_i^* \neq E_j$ then $\langle\langle E_i/E_j\rangle\rangle_\eta = 0$. Therefore, the η-orthogonality of the eigenvectors suggests that if ψ is an eigenvector (of eigenvalue $E = E_1 + iE_2$, $\forall E_1, E_2 \in \mathbb{R}$) related to H then

$$\eta\psi = 0 \quad \Longrightarrow \quad O^\dagger O\psi = 0 \quad \Longrightarrow \quad O\psi = 0, \tag{10}$$

and

$$Z(r) = -\frac{\psi'(r)}{\psi(r)} = -\partial_r \ln\psi(r) \quad \Longrightarrow \quad \psi(r) = \exp\left(-\int^r Z(z)dz\right). \tag{11}$$

Let us recast the linear operators' proposal in (6) as

$$\partial_r = O - Z(r) \quad \Longrightarrow \quad -\partial_r = O^\dagger - Z^*(r), \tag{12}$$

to imply

$$-\partial_r^2 = O^\dagger O - O^\dagger Z(r) - Z^*(r)O + Z^*(r)Z(r).$$

Hence, with $E = E_1 + iE_2$ and H in (5) the eigenvalue problem $H\psi(r) = E\psi(r)$ implies

$$\left[O^\dagger O - O^\dagger Z(r) - Z^*(r)O + Z^*(r)Z(r) + V_{\mathit{eff}}(r)\right]\psi(r) = E\psi(r). \tag{13}$$

This in turn, collapses into Riccati-type equation:

$$Z'(r) - Z^2(r) + V_{\mathit{eff}}(r) = E. \tag{14}$$

The real part of which reads

$$\operatorname{Re} V_{\mathit{eff}}(r) = -F'(r) + F(r)^2 - G(r)^2 + E_1, \tag{15}$$

and the imaginary part reads

$$\operatorname{Im} V_{\mathit{eff}}(r) = -G'(r) + 2F(r)G(r) + E_2. \tag{16}$$

On the other hand, the intertwining relation $H^\dagger\eta = \eta H$ along with (6) and (7) would lead to

$$\left[V^*_{\mathit{eff}}(r) - V_{\mathit{eff}}(r)\right] = -2M'(r) \quad\Longrightarrow\quad \operatorname{Im} V_{\mathit{eff}}(r) = -iM'(r), \tag{17}$$

$$2V'_{\mathit{eff}}(r) = M''(r) + 2N'(r) + \left[M(r)^2\right]' \tag{18}$$

and

$$-V''_{\mathit{eff}}(r) + M(r)V'_{\mathit{eff}}(r) = -N''(r) - 2M'(r)N(r) \tag{19}$$

to imply, respectively,

$$\operatorname{Im} V_{\mathit{eff}}(r) = -2G'(r), \tag{20}$$

$$\operatorname{Re} V_{\mathit{eff}}(r) = F(r)^2 - G(r)^2 - F'(r) + \beta \tag{21}$$

and

$$F(r)^2 - F'(r) = \frac{2G(r)G''(r) - G'(r)^2 + \alpha}{4G(r)^2}, \tag{22}$$

where $\alpha, \beta \in \mathbb{R}$ are integration constants. Eventually, with (21) and (15) implying

$$E_1 = \beta,$$

one would recast (15) as

$$\operatorname{Re} V_{\mathit{eff}}(r) = \left[\frac{2G(r)G''(r) - G'(r)^2 + \alpha}{4G(r)^2}\right] - G(r)^2 + \beta. \tag{23}$$

Moreover, equations (20) and (16) would imply

$$F(r) = -\frac{[G'(r) + E_2]}{2G(r)} \quad\Longrightarrow\quad f(r) = \frac{(\ell + 1)}{r} - \frac{[G'(r) + E_2]}{2G(r)}, \tag{24}$$

which when substituted in (22) yields

$$E_2^2 = \alpha.$$

Obviously, one would accept

$$\mathbb{R} \ni \alpha \geq 0 \quad \Longrightarrow \quad \mathbb{R} \ni E_2 = \pm\sqrt{\alpha} = \omega\sqrt{\alpha},$$

and negate

$$\alpha < 0 \quad \Longrightarrow \quad E_2 \in \mathbb{C}$$

since $\mathbb{R} \ni E_2 \notin \mathbb{C}$, as defined early on. Yet $E_2 \in \mathbb{C}$ contradicts with the real/imaginary descendants, (15) and (16), of the Riccati-type equation (14). Therefore, with $E_2 = \omega\sqrt{\alpha}$,

$$f(r) = \frac{(\ell+1)}{r} - \frac{[G'(r) + \omega\sqrt{\alpha}]}{2G(r)}, \tag{25}$$

and

$$\psi(r) = \sqrt{G(r)}\exp\left(\int^r \left[\frac{\omega\sqrt{\alpha}}{2G(z)} - iG(z)\right]dz\right), \tag{26}$$

where

$$G(r) = g(r)r^{2(\ell+1)}\left[\left(r^{n_r} + \sum_{p=0}^{n_r-1} A_{p,\ell}^{(n_r)} r^p\right)^2\right]. \tag{27}$$

Nevertheless, with $\alpha = 0$ one can express $G(r)$ in terms of $F(r)$, i.e.,

$$G(r) = \exp\left(-2\int^r F(z)dz\right) = r^{2(\ell+1)}\exp\left(-2\int^r f(z)dz\right). \tag{28}$$

Hence, $G(r)$ and/or $F(r)$ (equivalently $f(r)$ and/or $g(r)$, i.e. the generators reported by Fityo in [38]) can be considered as generating function(s) of the *η-weak-pseudo-Hermiticity* of non-Hermitian Hamiltonians with real spectra, where $\psi(r)$ in (26) is an eigen function of H in (5), but not necessarily normalizable. Moreover, the reality of the spectrum, in the forthcoming sections of our proposal, is secured by the choice $\mathbb{R} \ni \alpha = 0$. It should be noted here that the dependence of the radial wave function on the orbital angular momentum quantum number ℓ is obvious in (26) and (27). Yet, the formalism of the 1D case described by Fityo [38] may very well be reproduced by the trivial traditional setting $\ell = -1$, $n_r = 0$, and $r \to x \in (-\infty, \infty)$.

3 Illustrative Examples

In this section, we construct *η-weak-pseudo-Hermiticity* of some non-Hermitian Hamiltonians using the above mentioned procedure through the following illustrative examples.

3.1 Perturbed 1D-Harmonic Oscillator η-Weak-Pseudo-Hermiticity Generator(s)

In this section we consider the perturbed 1D-harmonic oscillator (i.e., $\ell = -1$, $r \to x \in (-\infty, \infty)$) with its real-valued generating-function $g(x) = \exp(-x^2)$ in (27) to get:

(A) For $n_r = 0$, the effective potential

$$V_{eff}(x) = x^2 - e^{-2x^2} + 4ixe^{-x^2} + \beta - 1, \tag{29}$$

which in turn leads to a $\mathcal{PT}$-symmetric *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$H = -\partial_x^2 + x^2 + e^{-x^2}\left(4ix - e^{-x^2}\right) + \beta - 1, \tag{30}$$

with a corresponding node-less (i.e., $n_r = 0$) wave function

$$\psi_0(x) = N_0 \exp\left(-\frac{1}{2}x^2 - \frac{1}{2}i\sqrt{\pi}\,\mathrm{erf}(x)\right). \tag{31}$$

It should be noted that $\psi_0(x) \sim e^{-x^2/2}$ in (31) represents the exact ground state wave function of the well known 1D-harmonic oscillator. The effective potential, on the other hand, represents a dominating real 1D-harmonic oscillator potential $V(x) = x^2$ perturbed by a weak interaction, with a real part $(-e^{-2x^2})$ and an imaginary part $(+4xe^{-x^2})$.

(B) For $n_r = 1$, and $A_{0,\ell}^{(1)} = 0$, the effective potential

$$V_{eff}(x) = x^2 - x^4 e^{-2x^2} + 4ixe^{-x^2}\left(x^2 - 1\right) + \beta - 3, \tag{32}$$

which in turn leads to an *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$H = -\partial_x^2 + x^2 - x^4 e^{-2x^2} + 4ixe^{-x^2}\left(x^2 - 1\right) + \beta - 3, \tag{33}$$

with a corresponding one-nodal (i.e., $n_r = 1$) wave function

$$\psi_1(x) = N_1 x \exp\left(-\frac{1}{2}x^2 + \frac{ix}{2}e^{-x^2} - \frac{1}{4}i\sqrt{\pi}\,\mathrm{erf}\,(x)\right). \tag{34}$$

It should be noted that $\psi_1(x) \sim xe^{-x^2/2}$ is the exact first-excited state wave function of the 1D-harmonic oscillator.

(C) For $n_r = 2$, $A_{0,\ell}^{(2)} = 0$, and $A_{1,\ell}^{(2)} = -1$ the effective potential

$$\begin{aligned} V_{eff}(x) &= x^2 - \frac{2}{x} - x^4(x-1)^4 e^{-2x^2} \\ &\quad + 4ixe^{-x^2}(x-1)\left(x^3 - x^2 - 2x + 1\right) + \beta - 5, \end{aligned} \tag{35}$$

which in turn leads to an *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$\begin{aligned} H &= -\partial_x^2 + x^2 - \frac{2}{x} - x^4(x-1)^4 e^{-2x^2} \\ &\quad + 4ixe^{-x^2}(x-1)(x^3 - x^2 - 2x + 1) + \beta - 5, \end{aligned} \tag{36}$$

with a corresponding two-nodal (i.e., $n_r = 2$) wave function

$$\begin{aligned} \psi_2(x) &= N_2 x(x-1) \exp\left(-\frac{1}{2}x^2\right) \\ &\quad \times \exp\left(ie^{-x^2}\left[\frac{x^3}{2} - x^2 + \frac{5x}{4} - 1\right] - \frac{5}{8}i\sqrt{\pi}\,\mathrm{erf}\,(x)\right). \end{aligned} \tag{37}$$

It should be noted that $\psi_1(x) \sim x(x-1)e^{-x^2/2}$ is the exact first-excited state wave function of the harmonic oscillator.

3.2 Perturbed Radial Harmonic Oscillator η-Weak-Pseudo-Hermiticity Generator(s)

With $\ell \geq 0$ and $r \in (0, \infty)$, we consider the radial harmonic oscillator generating function $g(r) = \exp(-r^2)$ to yield:

(A) For $n_r = 0$, the effective potential

$$\begin{aligned} \operatorname{Re} V_{\mathit{eff}}(r) &= \frac{\ell(\ell+1)}{r^2} + r^2 - r^{4(\ell+1)} e^{-2r^2} - 2(\ell+1) + \beta - 1, \\ \operatorname{Im} V_{\mathit{eff}}(r) &= 4r^{(2\ell+1)} \left[r^2 - (\ell+1) \right] e^{-r^2}, \end{aligned} \tag{38}$$

which in turn leads to an *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$H = -\partial_r^2 + \operatorname{Re} V_{\mathit{eff}}(r) + i \operatorname{Im} V_{\mathit{eff}}(r), \tag{39}$$

with a corresponding node-less (i.e., $n_r = 0$) wave function

$$\psi_{0,\ell}(r) = N_{0,\ell}\, r^{(\ell+1)} \exp\left(-\frac{1}{2} r^2 - i \int^r z^{2(\ell+1)} e^{-z^2} dz \right). \tag{40}$$

(B) For $n_r = 1$, with $A_{0,\ell}^{(1)} = 1$ in (27), the effective potential

$$\begin{aligned} \operatorname{Re} V_{\mathit{eff}}(r) = \frac{\ell(\ell+1)}{r^2} &+ r^2 - r^{4(\ell+1)} e^{-2r^2} (r-1)^4 \\ &+ \frac{2[\ell+1-r^2]}{r(r-1)} - 2(\ell+1) + \beta - 1, \end{aligned} \tag{41}$$

$$\operatorname{Im} V_{\mathit{eff}}(r) = 4(r-1)(r^3 - r^2 - (\ell+2)r + \ell + 1) r^{2\ell+1} \exp(-r^2), \tag{42}$$

which in turn leads to an *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$H = -\partial_r^2 + \operatorname{Re} V_{\mathit{eff}}(r) + i \operatorname{Im} V_{\mathit{eff}}(r), \tag{43}$$

with a corresponding one-nodal (i.e., $n_r = 1$) wave function

$$\begin{aligned} \psi_{1,\ell}(r) &= N_{1,\ell}\, r^{(\ell+1)} (r-1) \\ &\quad \times \exp\left(-\frac{1}{2} r^2 - i \int^r z^{2(\ell+1)} (z-1)^2 e^{-z^2} dz \right). \end{aligned} \tag{44}$$

3.3 Perturbed Radial Coulomb η-Weak-Pseudo-Hermiticity Generator(s)

With $\ell \geq 0$ and $r \in (0, \infty)$, let the real-valued function $g(r) = \exp(-2r)$ be substituted in (27), then

(A) For $n_r = 0$, the effective potential

$$\begin{aligned} \operatorname{Re} V_{\mathit{eff}}(r) &= \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} - r^{4(\ell+1)} e^{-4r} + \beta + 1, \\ \operatorname{Im} V_{\mathit{eff}}(r) &= 4(r - \ell - 1)\, r^{(2\ell+1)} e^{-2r}, \end{aligned} \tag{45}$$

which in turn leads to an *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$H = -\partial_r^2 + \mathrm{Re}\, V_{\mathit{eff}}(r) + i\, \mathrm{Im}\, V_{\mathit{eff}}(r), \tag{46}$$

with a corresponding node-less (i.e., $n_r = 0$) wave function

$$\psi_{0,\ell}(r) = N_{0,\ell}\, r^{(\ell+1)} \exp\left(-r - i \int^r z^{2(\ell+1)}\, e^{-2z} dz\right). \tag{47}$$

(B) For $n_r = 1$, and $A_{0,\ell}^{(1)} = 1$ in (27), the effective potential

$$\begin{aligned}\mathrm{Re}\, V_{\mathit{eff}}(r) = {} & \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} - r^{4(\ell+1)} e^{-4r} (r-1)^4 \\ & + \frac{2(\ell+1)}{r(r-1)} - \frac{2}{r-1} - 2(\ell+1) + \beta + 1,\end{aligned} \tag{48}$$

$$\mathrm{Im}\, V_{\mathit{eff}}(r) = 4(r-1)(r^2 - (\ell+3)r + \ell + 1) r^{2\ell+1} e^{-2r}, \tag{49}$$

which in turn leads to an *η-weak-pseudo-Hermitian Hamiltonian* of the form

$$H = -\partial_r^2 + \mathrm{Re}\, V_{\mathit{eff}}(r) + i\, \mathrm{Im}\, V_{\mathit{eff}}(r), \tag{50}$$

with a corresponding one-nodal (i.e., $n_r = 1$) wave function

$$\begin{aligned}\psi_{1,\ell}(r) = {} & N_{1,\ell}\, r^{(\ell+1)} (r-1) \\ & \times \exp\left(-r - i \int^r z^{2(\ell+1)} (z-1)^2\, e^{-2z} dz\right).\end{aligned} \tag{51}$$

4 Concluding Remarks

In this work, we have presented a class of spherically symmetric non-Hermitian Hamiltonians and their *η-weak-pseudo-Hermiticity generators. We have used* an operators-based procedure to come out with *η-weak-pseudo-Hermitian* non-Hermitian weakly perturbed 1D and radial harmonic oscillators, perturbed radial Coulomb, and the radial Morse models. We have presented not only nodeless 1D and radial wave functions but also some of the multi-nodal ones.

In the light of this experience, we have witnessed that the form of an *η-weak-pseudo-Hermitian Hamiltonian* changes for different values of ℓ and n_r. However, the reader should be reminded that our current proposal does not target exact solvability of *η-weak-pseudo-Hermitian Hamiltonians* but rather produces non-Hermitian *η-weak-pseudo-Hermitian Hamiltonians* with real spectra. Consequently, the Hermiticity requirement to ensure the reality of the spectrum of a Hamiltonian is shown to be not only fragile but also mathematically unnecessarily strong. Nevertheless, more physically oriented (although in 1D-case) applications of the above procedure could be found in our recent work in [48]. Therein, we have explored just a possibility of exact solvability through a Scarf II and a periodic-type *η-weak-pseudo-Hermitian Hamiltonian* models. Yet, another option for the intertwining operator η as a first-order intertwiner is explored in [49] with few informative examples presented.

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